ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, Nos. 1–2, 1985

A NONSEPARABLE BANACH SPACE NOT CONTAINING A SUBSYMMETRIC BASIC SEQUENCE

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ABSTRACT

We give an example of a nonseparable Banach space which does not contain a subsymmetric basic sequence. The space is the dual of a space constructed analogously to the James Tree space, using the Tsirelson space in place of l_2.

The first example of an infinite dimensional Banach space not containing a subsymmetric basic sequence was given by Tsirelson [16]. Tsirelson's space is separable and reflexive. Since [16] appeared, much more work has been done on this space, its dual and a number of variations thereof (see [3] for a thorough discussion). In particular Figiel and Johnson [5] have shown that there exists a space with a symmetric basis not containing an isomorph of c_0 or l_p $(1 \leqq p < \infty)$. Thus one immediately obtains the existence of a nonseparable space (with symmetric basis) which does not contain c_0 or l_p $(1 \leqq p < \infty)$.

In this paper we give an example of a nonseparable space, X, not containing a subsymmetric basic sequence. X is the dual of a separable space, Λ_T. Λ_T is defined analogously to the James Tree space [7] except that we use the unit vector basis of T_M (modified Tsirelson space, [8]) instead of l_2 in defining the norm. As we shall see, Λ_T does not contain l_1 (and so Λ_T^* does not contain c_0) and yet all spreading models of Λ_T^* are equivalent to either the summing basis or the unit vector basis of c_0. We shall also show that Λ_T^{**} is isomorphic to $\Lambda_T \oplus l_1(\Delta)$, where Δ is the Cantor set and $c_0(\Delta)$ is a quotient of Λ_T^*. These results have analogues in the James Tree space [10].

The problem which we solve in this paper was first brought to our attention several years ago by H. P. Rosenthal. Rosenthal and Shelah (unpublished)

[†]Research partially supported by NSF Grant No. DMS-8403669.
Received March 4, 1985

showed that there exists in some Banach space an uncountable normalized set, $(x_\alpha)_{\alpha\in\Gamma}$, so that every sequence of distinct x_α's is weakly null and not subsymmetric. In a positive direction, W. Henson has shown that this cannot happen in an L_1-space [6]. Indeed, he has shown there must exist a sequence $(x_{\alpha_i})_{i=1}^\infty$ which is almost exchangeable (see [1] for a discussion of almost exchangeable sequences) and in particular symmetric. Also, J. Ketonen [9] showed that if X is a Banach space of the cardinality of a Ramsey cardinal, then X contains a subsymmetric basic sequence.

We use standard Banach space terminology as may be found in [11].

1. Definition of Λ_T

Let T_M be the modified Tsirelson space discovered by Johnson [8] and let (e_n) be its unit vector basis. (e_n) is a 1-unconditional normalized basis for T_M satisfying for all $x \in T_M$ the implicit relation

$$(*)\qquad \|x\| = \sup\left\{\|x\|_{c_0}, \tfrac{1}{2}\sum_{i=1}^{n}\|E_i x\|\right\}$$

where the "sup" is taken over all finite collections of pairwise disjoint subsets of N, $(E_i)_{i=1}^n$, with $n \leqq \min E_i$ $(1 \leqq i \leqq n)$. If $E \subseteq \mathrm{N}$ and $x = \sum a_i e_i \in T_M$, $Ex = \sum_{i\in E} a_i e_i$. $(*)$ implies

(i) If $(x_i)_{i=1}^n$ are normalized vectors in T_M whose supports relative to $(e_i)_{i=1}^\infty$ are disjoint and contained in $\{e_n, e_{n+1}, \ldots\}$, then $(x_i)_{i=1}^n$ is 2-equivalent to the unit vector basis of l_1^n.

T_M is reflexive [8]. Moreover we shall need the following fact. There exists $c < \infty$ so that if (x_i) is a normalized block basis of (e_i) with $x_n = \sum_{i=l_n+1}^{l_{n+1}} a_i e_i$, then for any choice of integers $l_n < k_n \leqq l_{n+1}$ and scalars (b_n),

$$\text{(ii)}\qquad c^{-1}\left\|\sum b_n e_{k_n}\right\| \leqq \left\|\sum b_n x_n\right\| \leqq c\left\|\sum b_n e_{k_n}\right\|.$$

This was proved in [4] for the space T (the dual of the original Tsirelson example [5]) and the same argument works for T_M. In fact this was used to prove T and T_M are naturally isomorphic [2].

Let Λ denote the infinite dyadic tree,

$$\Lambda = \{(n,i): n = 0,1,2,\ldots, 1 \leqq i \leqq 2^n\}.$$

The *nth level* of Λ is $\{(n,i): 1 \leqq i \leqq 2^n\}$. Λ is partially ordered by putting $(n,i) \leqq (m,j)$ if $K_{n,i} \supseteq K_{m,j}$ where $(K_{n,i})$ are the triadic intervals used in forming the Cantor set. Thus $K_{0,1} = [0,1]$, $K_{1,1} = [0,\frac{1}{3}]$, $K_{1,2} = [\frac{2}{3},1]$, etc. By a *segment* β, in Λ we mean a linearly ordered set of the form

$$\{(n,i_1),(n+1,i_2),(n+2,i_3),\dots\},$$

either finite or infinite in length. In particular β could be a single element of Λ.

We shall also have use for a linear order on Λ. We define $d(0,1)=1$ and $d(n,i)=\sum_{j=0}^{n-1}2^j+i$ if $n \geqq 1$. If β is a segment with initial node (n,i), we define its *order* by $o(\beta)=d(n,i)$.

Let x be a real valued function defined on Λ with finite support. If β is a segment in Λ, set

$$S_\beta(x)=\sum_{(n,i)\in\beta} x(n,i).$$

The norm of x in Λ_T is defined by

$$\|x\|=\sup\left\{\left\|\sum_{i=1}^{n} S_{\beta_i}(x)e_{o(\beta_i)}\right\|_{T_M} : (\beta_i)_{i=1}^{n} \text{ are pairwise disjoint segments in } \Lambda\right\}.$$

We call $y=\sum_{i=1}^{n} S_{\beta_i}(x)e_{o(\beta_i)}$ (with $(\beta_i)_1^n$ disjoint segments) a *representative* of x in T_M. Λ_T is defined to be the completion under this norm.

In the James Tree space the norm is defined exactly the same, except that T_M is replaced by l_2 and the e_i's are the unit vector basis of l_2. Since this basis is symmetric, one may use e_i in place of $e_{o(\beta_i)}$ in the definition.

We note that by the definition of the norm in Λ_T if β is any nonempty segment (finite or infinite), then S_β extends naturally to a norm one functional in Λ_T^*. Also if $(x_{(n,i)})_{(n,i)\in\Lambda}$ are the unit node vectors in Λ_T,

$$x_{(n,i)}(m,j)=\delta_{((n,i),(m,j))},$$

then $(x_{(n,i)})_{(n,i)\in\Lambda}$ forms a normalized monotone basis for Λ_T under the ordering induced by $d(n,i)$. Indeed, the projections P_d given by

$$P_d\left(\sum_{(n,i)\in\Lambda} a_{n,i}x_{(n,i)}\right)=\sum_{\{(n,i):d(n,i)\leqq d\}} a_{n,i}x_{(n,i)}$$

are norm one for all $d \in \mathbf{N}$.

To get a feel for the norm in Λ_T we give a very easy and useful lemma. Roughly speaking, the lemma says that if $x\in\Lambda_T$ is supported on the levels of Λ greater than or equal to the nth, then $\|x\|$ may be calculated (up to a constant) by using in the definition only segments which originate on the nth or greater levels.

LEMMA 1. *For all $n \in \mathbf{N}$ and $x \in \Lambda_T$ with $P_{2^n-1}x=0$,*

$$4^{-1}\|x\| \leqq \sup\{\|\textstyle\sum_{i=1}^k S_{\beta_i}(x)e_{o(\beta_i)}\|_{T_M} : (\beta_i)_1^k \text{ are disjoint segments in } \Lambda \text{ with } o(\beta_i) \geqq 2^n \text{ for } 1 \leqq i \leqq k\}.$$

PROOF. Let $1 = \|x\| = \|\sum_{i=1}^k S_{\beta_i}(x)e_{o(\beta_i)}\|_{T_M}$, where $(\beta_i)_1^k$ are disjoint segments in Λ. We may suppose there is an l, $0 \leqq l \leqq k$, so that for $i > l$, $o(\beta_i) \geqq 2^n$ and for $i \leqq l$, $o(\beta_i) < 2^n$. If

$$\left\| \sum_{i=l+1}^k S_{\beta_i}(x)e_{o(\beta_i)} \right\|_{T_M} \geqq \tfrac{1}{2},$$

fine. If not, then

$$\left\| \sum_{i=1}^l S_{\beta_i}(x)e_{o(\beta_i)} \right\|_{T_M} \geqq \tfrac{1}{2}.$$

For $i \leqq l$, let $\bar{\beta}_i$ be the largest segment contained in β_i with initial node of the form (n, j_i). Since $P_{2^n-1}x = 0$, $S_{\bar{\beta}_i}(x) = S_{\beta_i}(x)$. Also, since $l \leqq 2^n - 1$ and $o(\bar{\beta}_i) \geqq 2^n$ $(1 \leqq i \leqq l)$, by (i) we have

$$\left\| \sum_{i=1}^l S_{\bar{\beta}_i}(x)e_{o(\bar{\beta}_i)} \right\|_{T_M} \geqq 2^{-1} \sum_{i=1}^l |S_{\bar{\beta}_i}(x)|$$

$$\geqq 2^{-1} \left\| \sum_{i=1}^l S_{\beta_i}(x)e_{o(\beta_i)} \right\| \geqq 4^{-1}. \qquad \blacksquare$$

Before stating our main result, we recall the notion of a spreading model. Let (x_n) be a bounded basic sequence in a Banach space. A (necessarily subsymmetric) basic sequence, (y_n), in another Banach space is said to be a *spreading model* for (x_n) if for all scalars $(a_i)_{i=1}^k$,

$$\lim_{\substack{n_1<n_2<\cdots<n_k \\ n_1\to\infty}} \left\| \sum_{i=1}^k a_i x_{n_i} \right\| = \left\| \sum_{i=1}^k a_i y_i \right\|.$$

The Brunel–Sucheston theorem (see e.g., [12]) states that if (x_n) is a bounded basic sequence with no norm convergent subsequence, then some subsequence, (x'_n), has a spreading model (y_n). Furthermore, if (x'_n) is weakly null, then (y_n) is an unconditional basic sequence.

In this language, T_M has the property that all of its spreading models are equivalent to the unit vector basis of l_1.

2. The main theorem

THEOREM 2.

(1) *The vectors* $(x_{(n,i)})_{(n,i)\in\Lambda}$ *form a boundedly complete monotone basis for* Λ_T.

(2) *All spreading models in Λ_T are equivalent to the unit vector basis of l_1.*

(3) *Λ_T^* is nonseparable.*

(4) *Λ_T does not contain an isomorph of l_1 and so Λ_T^* does not contain an isomorph of c_0.*

(5) *Λ_T^* is the closed linear span of $\{S_\beta : \beta$ is a segment in $\Lambda\}$.*

(6) *All spreading models in Λ_T^* are equivalent to either the summing basis or the unit vector basis of c_0.*

(7) *$c_0(\Delta)$ is a quotient of Λ_T^*, where Δ is the Cantor set.*

(8) *Λ_T^{**} is isomorphic to $\Lambda_T \oplus l_1(\Delta)$.*

PROOF. (1) For $n < m$, let $Q_{n,m} = P_{2^m-1} - P_{2^n-1}$. To prove (1) it suffices to show

LEMMA 3. *Let $(x_i)_{i=1}^k$ be normalized vectors in Λ_T. Assume there exist integers $n_0 < n_1 < \cdots < n_k$ so that for $1 \leqq i \leqq k$, $x_i = Q_{n_{i-1},n_i} x_i$. Then if $k \leqq 2^{n_0}$, $(x_i)_{i=1}^k$ is 8-equivalent to the unit vector basis of l_1^k.*

PROOF. By (i) it suffices to show that for $1 \leqq i \leqq k$, there are disjoint segments $(\beta_j^i)_{j=1}^{p(i)}$ in Λ so that each β_j^i lies between the n_{i-1} and n_i levels of Λ and the representatives y_i of x_i given by $y_i = \sum_{j=1}^{p(i)} S_{\beta_j^i}(x_i) e_{o(\beta_j^i)}$ satisfy $\| y_i \| \geqq 4^{-1} \| x_i \|$. This is easily done by Lemma 1. ∎

We shall prove a stronger result later in Lemma 10.

(2) This is a corollary of (1) and Lemma 3. Indeed, every basic sequence $(y_i) \subseteq \Lambda_T$ with a spreading model has a subsequence essentially (up to a perturbation) of the form $x + x_i$ for some $x \in \Lambda_T$, $(x_i) \subseteq \Lambda_T$, where $0 < \inf_i \| x_i \| \leqq \sup_i \| x_i \| < \infty$ and $Q_{n_{i-1},n_i} x_i = x_i$ for some $(n_i) \subseteq \mathbf{N}$.

(3) This is trivial. If $\beta \neq \gamma$ are infinite segments and $(n,i) \in \beta \setminus \gamma$ then $\| S_\beta - S_\gamma \| \geqq (S_\beta - S_\gamma)(x_{(n,i)}) = 1$.

(4) This is a bit more complicated. Part of the proof is similar to an argument in [13]. Suppose Λ_T contains l_1. Then there exists a normalized sequence $(x_i) \subseteq \Lambda_T$ which is equivalent to the unit vector basis of l_1 and satisfies $Q_{n_{i-1},n_i} x_i = x_i$ for some increasing sequence of integers, (n_i).

For each element $x \in \Lambda_T$ we associate an element $\hat{x} \in C(\Delta)$ as follows. We identify the points of Δ with the set of maximal segments (branches) of Λ in the natural way. Thus if $\beta = \{(0, i_1), (1, i_2), \ldots\}$ is a branch, β may be regarded as the point in Δ given by $\bigcap_{n=0}^\infty K_{n, i_{n+1}}$. We set $\hat{x}(\beta) = S_\beta(x)$.

LEMMA 4. *The map $x \to \hat{x}$ is a norm one linear mapping of Λ_T into $C(\Delta)$.*

PROOF. For $\beta \in \Delta$, $|\hat{x}(\beta)| = |S_\beta(x)| \leqq \|x\|$. All that needs to be checked is that $\hat{x}$ is continuous. But if $\beta_n \to \beta$ in Δ then $S_{\beta_n}(x) \to S_\beta(x)$, or else we would have a sequence of disjoint finite segments, (γ_n), in Λ with $|S_{\gamma_n}(x)| \geqq \varepsilon > 0$ for some $\varepsilon > 0$ and all $n \in \mathbf{N}$, in which case $\|x\| = \infty$. ■

We first show that $(\hat{x}_i)$ cannot be equivalent in $C(\Delta)$ to the unit vector basis of l_1. For if $(\hat{x}_i)$ is equivalent to the unit vector basis of l_1, then by Rosenthal's theorem [15] we may assume (by passing to a subsequence and relabeling) that there exist $r \in \mathbf{R}$ and $\delta > 0$ so that if $A_i = \{t \in \Delta : \hat{x}_i(t) > r + \delta\}$ and $B_i = \{t \in \Delta : \hat{x}_i(t) < r\}$, then (A_i, B_i) is Boolean independent. This means that for $k \in \mathbf{N}$ and all $\varepsilon = (\varepsilon_i)_{i=1}^k$ with $\varepsilon_i = \pm 1$, the set 0_ε is nonempty where $0_\varepsilon = \bigcap_{i=1}^k \varepsilon_i A_i$ ($\varepsilon A_i = A_i$ if $\varepsilon = 1$ and $\varepsilon A_i = B_i$ if $\varepsilon = -1$). The 0_ε's thus comprise 2^k disjoint open sets in Δ. We may assume $r + \delta > 0$ (if not replace (x_i) by $(-x_i)$).

Choose m_0 so large that for $1 \leqq i \leqq k$ and $1 \leqq j \leqq 2^{m_0}$, the oscillation of each of the continuous functions $\hat{x}_i$ on $K_{m_0,j} \cap \Delta$ is less than $\delta/2$. It follows that for all j, $K_{m_0,j} \cap 0_\varepsilon \neq \emptyset$ for at most one ε. Choose $i_0 > k$ so that $P_{2^{m_0}-1} x_{i_0} = 0$. Then for all $\varepsilon = (\varepsilon_i)_1^k$, $0_\varepsilon \cap A_{i_0} \neq \emptyset$ and this implies $\|\hat{x}_{i_0}|_{K_{m_0,j} \cap \Delta}\|_\infty > r + \delta > 0$ on at least 2^k distinct $K_{m_0,j}$'s. Thus $S_{\beta_j}(x_{i_0}) > r + \delta$ for at least 2^k disjoint segments, $(\beta_j)_{j=1}^{2^k}$, with $O(\beta_j) \geqq 2^{m_0} \geqq 2^k$, and so by (i), $\|x_{i_0}\| \geqq 2^{k-1}(r + \delta)$, which is impossible for large enough k.

Thus we may assume by Rosenthal's theorem [15] that $(\hat{x}_i)$ is pointwise convergent in $C(\Delta)$. By taking differences and then far out convex combinations we may assume that our l_1 basis, (x_i), also satisfies $\|\hat{x}_i\|_\infty \to 0$. But this is impossible as the following lemma shows.

LEMMA 5. *Let (x_i) be a normalized block basis of $(x_{(n,i)})$ with $\|\hat{x}_i\|_\infty \to 0$. Then there exists a block basis of convex combinations of (x_i) which is equivalent to some subsequence (e_{p_i}) of (e_i) in T_M.*

PROOF. By Lemma 3 and the hypothesis there exists (y_i), a block basis of convex combinations (actually long averages) of (x_i) with $1 \geqq \|y_i\| \geqq 8^{-1}$ for all i and

$$\sup\{|S_\beta(y_i)| : \beta \text{ is a segment}\} = \varepsilon_i$$

where $\varepsilon_i \to 0$ as $i \to \infty$. We may also assume

$$Q_{m_{2i-1}, m_{2i}} y_i = y_i \text{ for some increasing sequence } (m_i) \subseteq \mathbf{N},$$

and if $p_i = d(m_{2i}, 1)$, then

(iii) $$\sum_{j>i} \varepsilon_j < 2^{-(p_i+i)}.$$

We shall prove that (y_i) is equivalent to (e_{p_i}). We first show that (e_{p_i}) dominates (y_i).

Let $(a_i)_1^k$ be scalars and suppose (β_l) are disjoint segments in Λ with

$$\left\| \sum_{i=1}^{k} a_i y_i \right\| = \left\| \sum_{l} \left| S_{\beta_l}\left(\sum_i a_i y_i \right) \right| e_{o(\beta_l)} \right\|_{T_M}.$$

Let $F_i = \{l : p_{i-1} < o(\beta_l) \leqq p_i\}$, $1 \leqq i \leqq k$. Fix i and $l \in F_i$. Then

$$\begin{aligned} \left| S_{\beta_l}\left(\sum_{j=1}^{k} a_j y_j \right) \right| &\leqq | a_i S_{\beta_l}(y_i) | + \sum_{j>i} | a_j | \, | S_{\beta_l}(y_j) | \\ &\leqq | a_i | \, | S_{\beta_l}(y_i) | + \max_{j>i} | a_j | \sum_{j>i} \varepsilon_j. \end{aligned}$$

Thus by the 1-unconditionality of (e_i),

$$\text{(iv)} \qquad \left\| \sum_{i=1}^{k} a_i y_i \right\| \leqq \left\| \sum_{i=1}^{k} \sum_{l \in F_i} | a_i S_{\beta_l}(y_i) | e_{o(\beta_l)} \right\| + \left\| \sum_{i=1}^{k} \sum_{l \in F_i} \max_j | a_j | \sum_{j>i} \varepsilon_j e_{o(\beta_l)} \right\|.$$

Since for each $i \leqq k$,

$$\left\| \sum_{l \in F_i} | a_i S_{\beta_l}(y_i) | e_{o(\beta_l)} \right\| \leqq | a_i | \, \| y_i \| \leqq | a_i |,$$

and since $p_{i-1} < o(\beta_l) \leqq p_i$ for $l \in F_i$, by (ii) the first term on the right side of (iv) is

$$\leqq c \left\| \sum_{i=1}^{k} a_i e_{p_i} \right\|.$$

Also,

$$\begin{aligned} \left| \sum_{i=1}^{k} \sum_{l \in F_i} \sum_{j>i} \varepsilon_j \right| &\leqq \sum_{i=1}^{k} | F_i | \sum_{j>i} \varepsilon_j \\ &\leqq \sum_{i=1}^{k} 2^{p_i} \left(\sum_{j>i} \varepsilon_j \right) \leqq 2^{-i} \end{aligned}$$

by (iii). Thus the second term in (iv) is

$$\leqq \max_j | a_j | \sum_{i=1}^{\infty} 2^{-i} = \max_j | a_j |.$$

This proves (e_{p_i}) dominates (y_i).

By (ii) and Lemma 1 it follows that (y_i) dominates (e_{q_i}) where $q_i = d(m_{2i-1}, 1)$ and hence since (e_{q_i}) is equivalent to (e_{p_i}) (again, by (ii)), (y_i) is equivalent to (e_{p_i}). ∎

REMARK. We do not know if every infinite dimensional subspace of Λ_T contains a sequence equivalent to some (e_{p_i}).

The fact that Λ_T^* does not contain c_0 follows from Λ_T not containing l_1 by a classical result of Bessaga and Pelczynski (see [11], p. 103).

(5) We first prove

LEMMA 6. *Let (x_i) be a normalized block basis of $(x_{(n,i)})$. If $\hat{x}_i \to 0$ weakly in $C(\Delta)$, then $x_i \to 0$ weakly in Λ_T.*

PROOF. If (x_i) is not weakly null, then by passing to a subsequence we may suppose there is an $f \in \Lambda_T^*$, $\|f\| = 1$ and $f(x_i) \geqq \delta > 0$ for all i. Thus $\|y\| \geqq \delta$ whenever y is a convex combination of the x_i's. But there is a block basis, (y_i), of convex combinations of (x_i), with $\|\hat{y}_i\|_\infty \to 0$. Thus by Lemma 5, there is a block basis (z_i) of convex combinations of (x_i) which is weakly null, a contradiction. ∎

Let $[(S_\beta)]$ be the closed linear span of all S_β's where β is a segment. If $[(S_\beta)] \neq \Lambda_T^*$, there exists $F \neq 0$ in Λ_T^{**} with $F|_{[(S_\beta)]} = 0$. Since l_1 does not embed in Λ_T, by [14] there exists $(x_n) \subseteq \Lambda_T$ which converges weak* (in Λ_T^{**}) to F with $\|x_n\| = \|F\|$ for all n. Since $F|_{[(S_\beta)]} = 0$, we may suppose (x_n) is a block basis of $(x_{(n,i)})$. Thus since $\lim_n \hat{x}_n(\beta) = \lim_n S_\beta(x_n) = F(S_\beta) = 0$ for all segments β, by Lemma 6, (x_n) is weakly null and so $F = 0$, a contradiction.

(6) We shall show that if $(f_i) \subseteq \Lambda_T^*$ is a basic sequence with a spreading model, then $(f_{2i} - f_{2i+1})$ has a spreading model equivalent to the unit vector basis of c_0. (6) follows by the following lemma.

LEMMA 7. *Let (y_n) be a normalized subsymmetric basic sequence so that $(y_{2n} - y_{2n+1})$ is equivalent to the unit vector basis of c_0. Then (y_n) is equivalent to either the unit vector basis of c_0 or to the summing basis.*

REMARK. Since the spreading model of a weakly null sequence is unconditional, it will follow that the spreading model of every weakly null sequence in Λ_T is equivalent to the unit vector basis of c_0.

PROOF OF LEMMA 7. Recall that the summing basis, (s_n), satisfies $\|\Sigma a_i s_i\| = \sup_n |\Sigma_{i=1}^n a_i|$. If (y_n) is weakly null, it is unconditional hence there exists $K < \infty$ so that $\|\Sigma_1^n \varepsilon_i y_i\| \leqq K$ for all $\varepsilon_i = \pm 1$, $n \in \mathbf{N}$. Thus (y_n) is equivalent to the unit vector basis of c_0.

If (y_n) is not weakly null, then there exists $f \in [(y_n)]^*$ so that $f(y_{n_i}) \geqq \varepsilon > 0$ for some subsequence (y_{n_i}). Thus by subsymmetry, (y_n) dominates the summing basis. That (y_n) is dominated by (s_n) follows from the fact that $(y_{2n} - y_{2n+1})$ and

$(y_{2n-1} - y_{2n})$ are both K-equivalent to the unit vector basis of c_0, for some K. Indeed,

$$\sum a_i y_i = a_1(y_1 - y_2) + (a_1 + a_2)(y_2 - y_3) + (a_1 + a_2 + a_3)(y_3 - y_4)$$
$$+ (a_1 + a_2 + a_3 + a_4)(y_4 - y_5) + \cdots$$

and so

$$\left\| \sum a_i y_i \right\| \leqq 2K \left\| a_i s_i \right\|. \qquad \blacksquare$$

Before attacking the general $(f_i) \subseteq \Lambda_T^*$, we prove the result in a special case. Let $(x^*_{(n,i)})$ be the biorthogonal functionals to $(x_{(n,i)})$.

Lemma 8. *Let $(f_i)_{i=1}^k$ be a normalized block basis of $(x^*_{(n,i)})$ with*

$$f_i \in \operatorname{span}\{x^*_{(n,i)} : 2^{n_{i-1}} \leqq d(n,i) < 2^{n_i}\}$$

for $1 \leqq i \leqq k$ and integers $n_0 < n_1 < \cdots < n_k$. If $k \leqq 2^{n_0}$, then $(f_i)_{i=1}^k$ is 32-equivalent to the unit vector basis of l_∞^k.

Proof. Let $\|\Sigma_1^k a_i f_i\| = \Sigma_1^k a_i f_i(x)$ where $x \in \Lambda_T$, $\|x\| \leqq 2$ and $P_{2^{n_0}-1}x = 0$. Write $x = \Sigma_{i=1}^k x_i$ where $x_i = Q_{n_{i-1},n_i}x$. Then

$$\sum_{i=1}^k a_i f_i(x) = \sum_1^k a_i f_i(x_i)$$

$$\leqq \max_i |a_i| \max_i \|f_i\| \sum_1^k \|x_i\|$$

$$\leqq 8 \max_i |a_i| \, \|x\|$$

$$\leqq 16 \max_i |a_i|$$

(by Lemma 3).

Also if $\|x\| = 1$ and $f_{i_0}(x) = \operatorname{sign} a_{i_0}$,

$$\left\| \sum a_i f_i \right\| \geqq 2^{-1} \sum a_i f_i (Q_{i_0-1,i_0} x)$$
$$= 2^{-1} |a_{i_0}|. \qquad \blacksquare$$

For the general case, let (f_i) be a normalized basic sequence in Λ_T^* with a spreading model. We need only show $(f'_{2i} - f'_{2i+1})$ has a spreading model equivalent to the unit vector basis of c_0 for some subsequence, (f'_i).

We may thus assume $\lim_{j\to\infty} f_j(x_{(n,i)})$ exists for all $(n,i)\in\Lambda$. Also we may assume (by (5)) that $f_i \in \operatorname{span}\{S_\beta : \beta$ is a segment$\}$ for all i. The following lemma is an easy consequence of Lemma 6.

LEMMA 9. *Let β be an infinite segment in Λ. Then $(x_{(n,i)})_{(n,i)\in\beta}$ is weak Cauchy.*

PROOF. If not, there exists a subsequence (y_n) of $(x_{(n,i)})_{(n,i)\in\beta}$, an $f\in\Lambda_T^*$ and an $\varepsilon>0$ with $f(y_{2n}-y_{2n+1})>\varepsilon$ for all n. But $(\widehat{y_{2n}-y_{2n+1}})$ is weakly null in $C(\Delta)$ and so $(y_{2n}-y_{2n+1})$ is weakly null in Λ_T, a contradiction. ■

Write $f_1-f_2=g_1+h_1$ where $g_1\in\operatorname{span}\{x^*_{(n,i)}: d(n,i)<2^{m_1}\}$ and $h_1=\sum_{i=1}^{k(1)} a_i^1 S_{\beta_i^1}$, where the β_i^1's are pairwise disjoint infinite segments originating at the m_1-level of Λ (i.e., $2^{m_1}\leqq o(\beta_i^1)<2^{m_1+1}$ for $1\leqq i\leqq k(1)$).

Let

$$A_p(i_0)=\lim_{(n,i)\in\beta^1_{i_0}} f_p(x_{(n,i)}).$$

We may assume (by passing to a subsequence) that $A(i_0)=\lim_{p\to\infty}A_p(i_0)$ exists for $1\leqq i_0\leqq k(1)$. Thus we may assume (by perturbing the f_i's, if necessary) that for $j>2$,

$$f_j=\sum_{i=1}^{k(1)} A(i)S_{\beta_i^1}+d_j$$

where $d_j\in\operatorname{span}\{S_\beta:\beta$ is a segment in Λ, $\beta\cap\beta_i^1$ is finite for $1\leqq i\leqq k(1)\}$. Hence, by perturbing, we may suppose $f_3-f_4=g_2+h_2$ where $g_2\in\operatorname{span}\{x^*_{(n,i)}:2^{m_1}\leqq d(n,i)<2^{m_2}\}$ and $h_2=\sum_{i=1}^{k(2)} a_i^2 S_{\beta_i^2}$, where the β_i^2's are disjoint infinite segments originating at the m_2-level of Λ and moreover the β_i^2's are disjoint from the β_i^1's as well!

Continue in this fashion, obtaining (after passing to subsequences and perturbing) $f_{2l-1}-f_{2l}=g_l+h_l$ with $g_l\in\operatorname{span}\{x^*_{(n,i)}:2^{m_{l-1}}\leqq d(n,i)<2^{m_l}\}$ and $h_l=\sum_{i=1}^{k(l)} a_i^l S_{\beta_i^l}$ with the β_i^l's infinite pairwise disjoint segments originating on the m_l-level of Λ and disjoint from the β_i^j's for $1\leqq j<l$ and $1\leqq i\leqq k(j)$. Note that $\|g_i\|\leqq 2$ and $\|h_i\|\leqq 4$ for all i.

We claim that this sequence of differences of a subsequence of (f_i), which we have relabelled $(f_{2i}-f_{2i+1})$, has a spreading model equivalent to the unit vector basis of c_0. Indeed let $n\in\mathbb{N}$ and let $(a_i)_{i=n}^{2n}$ be scalars. We shall show

$$\text{(v)}\qquad \left\|\sum_{n}^{2n} a_i(f_{2i}-f_{2i+1})\right\|\leqq 416\max_i|a_i|,$$

which will complete the proof.

To see this we write

$$\left\|\sum_n^{2n} a_i(f_{2i}-f_{2i+1})\right\| \leqq \left\|\sum_n^{2n} a_i g_i\right\| + \left\|\sum_n^{2n} a_i h_i\right\|.$$

By Lemma 8,

$$\left\|\sum_n^{2n} a_i g_i\right\| \leqq 16 \max_i |a_i| \, \|g_i\| \leqq 32 \max_i |a_i|.$$

Let $\|x\| \leqq 2$ with $P_{2^{m_n}-1}x = 0$ and $\|\Sigma_n^{2n} a_i h_i\| = \Sigma_n^{2n} a_i h_i(x)$. For $n \leqq i < 2n$, let

$$E_i = \left\{(q,j) \in \Lambda : (q,j) \notin \bigcup_{n \leqq l < i} \bigcup_{p=1}^{k(l)} \beta_p^l \text{ and } 2^{m_i} \leqq o(q,j) < 2^{m_{i+1}}, \text{ or } (q,j) \in \bigcup_{p=1}^{k(i)} \beta_p^i\right\}.$$

Define

$$E_{2n} = \left\{(q,h) \in \Lambda : (q,j) \notin \bigcup_{n \leqq l < 2n} \bigcup_{p=1}^{k(l)} \beta_p^l \quad \text{and} \quad 2^{m_{2n}} \leqq O(q,j)\right\}.$$

Let

$$x = \sum_n^{2n} x_i \qquad \text{where } x_i \in \operatorname{span}\{x_{(q,j)} : (q,j) \in E_i\}.$$

Lemma 10. $\Sigma_n^{2n} \|x_i\| \leqq 48 \|x\|$.

Proof. It suffices to show that each x_i has a representative $y_i = \Sigma_j S_{\gamma_j^i}(x_i) e_{o(\gamma_j^i)}$ where the γ_j^i's are segments contained in E_i and $\|y_i\| \geqq (24)^{-1}\|x_i\|$.

Each x_i can be expressed as $x_i = \Sigma_{j=1}^{q(i)} x_j^i$ where $q(i) \leqq 2^{m_i}$ and the x_j^i's are disjointly supported vectors, each supported in E_i and "separated" from one another by the infinite branches — the β_p^l's for $l < i$.

Sublemma. *Let γ_1 and γ_2 be disjoint infinite segments, $\gamma_i = \{(m, j_0^i), (m+1, j_1^i), \ldots\}$ for $i = 1,2$, with $j_0^1 < j_0^2$. Let $(\beta_l)_{l=1}^r$ be disjoint infinite segments originating at the $(m+k)$-level of Λ $(k > 0)$ and suppose that $j_k^1 < j < j_k^2$ for all $(m+k, j) \in \bigcup_1^r \beta_l$. Let $F = \{(m+n, j) \in \Lambda : 0 \leqq n \leqq k$ and $j_n^1 < j < j_n^2\} \cup \bigcup_{l=1}^r \beta_l$. Let $x \in \operatorname{span}\{x_{(n,i)} : (n,i) \in F\}$. Then x has a representative, $y = \Sigma S_{\delta_i}(x) e_{o(\delta_i)}$, where the δ_i's are disjoint segments contained within F and $\|y\| \geqq 12^{-1}\|x\|$.*

Proof of Sublemma. Let $z = \Sigma S_{\alpha_i}(x) e_{o(\alpha_i)}$ be a representative of x with $\|z\| \geqq 4^{-1}\|x\|$, such that each segment α_i originates at level m or a larger level. We may assume *for all i, $\alpha_i \subseteq F \cup \gamma_1 \cup \gamma_2$ and* $S_{\alpha_i}(x) \neq 0$. Let $I_1 = \{i : \alpha_i$ originates on $\gamma_1\}$, $I_2 = \{i : \alpha_i$ originates on $\gamma_2\}$ and $I_3 = \{i : \alpha_i$ originates on $F\}$. Then for some $p = 1,2$ *or* 3

$$\left\|\sum_{i \in I_p} S_{\alpha_i} e_{o(\alpha_i)}\right\| \geqq 3^{-1}\|z\|.$$

If $p = 3$, we let $y = \sum_{i \in I_3} S_{\alpha_i}(x) e_{o(\alpha_i)}$. If $p = 1$ (a similar argument works for $p = 2$), for $i \in I_1$ let $\delta_i = \alpha_i \cap F$, and set $y = \sum_{i \in I_1} S_{\delta_i}(x) e_{o(\delta_i)}$. Since $S_{\alpha_i}(x) = S_{\delta_i}(x)$ and $o(\alpha_i) < o(\alpha_j)$ implies $o(\delta_i) < o(\delta_j)$,

$$3^{-1} \| z \| \leqq \left\| \sum_{i \in I_1} S_{\alpha_i}(x) e_{o(\alpha_i)} \right\| \leqq \| y \|.$$

This proves the sublemma. ■

Returning to the proof of Lemma 10, by applying the sublemma to each x_j^i we can find a representative y_i of x_i determined by segments contained wholly within E_i and with

$$\| y_i \| \geqq 2^{-1} \left(\sum_{j=1}^{q(i)} 12^{-1} \| x_j^i \| \right) \geqq (24)^{-1} \| x_i \|.$$ ■

Finally we complete the proof of (v):

$$\begin{aligned} \left\| \sum_n^{2n} a_i h_i \right\| &= \sum_n^{2n} a_i h_i(x) = \sum_n^{2n} a_i h_i(x_i) \\ &\leqq \max_i |a_i| \, \| h_i \| \sum_n^{2n} \| x_i \| \\ &\leqq 4(48) \| x \| \max_i |a_i| \qquad \text{by Lemma 10,} \\ &\leqq 384 \max_i |a_i|. \end{aligned}$$

The proofs of (7) and (8) are similar to arguments in [10].

(7) First we prove

Lemma 11. *Let $(\beta_i)_{i=1}^n$ be disjoint infinite segments in Λ, all originating at level m with $m \geqq n$. Then $(S_{\beta_i})_{i=1}^n$ is 2-equivalent to the unit basis of l_∞^n.*

Proof. Let $(a_i)_{i=1}^n$ be scalars and choose $x \in \Lambda_T$ with $\| x \| = 1$ and $\| \sum_1^n a_i S_{\beta_i} \| = \sum_1^n a_i S_{\beta_i}(x)$. Then clearly

$$\begin{aligned} \max_i |a_i| \leqq \left\| \sum_1^n a_i S_{\beta_i} \right\| &= \sum_1^n a_i S_{\beta_i}(x) \leqq \max_i |a_i| \sum_1^n |S_{\beta_i}(x)| \\ &\leqq 2 \max_i |a_i| \left\| \sum_1^n S_{\beta_i}(x) e_{o(\beta_i)} \right\| \\ &\leqq 2 \max_i |a_i|. \end{aligned}$$ ■

We claim that $\Lambda_T^*/[(x^*_{(n,i)})_{(n,i)\in\Lambda}]$ is isomorphic to $c_0(\Delta)$. Indeed, define $Q:\Lambda_T^*\to c_0(\Delta)$ by $Q(f)(\beta)=\lim_{(n,i)\in\beta}f(x_{(n,i)})$, the limit existing by Lemma 9. Q is a well defined bounded linear mapping with kernel $=[(x^*_{(n,i)}):(n,i)\in\Lambda]$ by (5) and Lemma 11.

(8) Since $(x_{(n,i)})$ is boundedly complete, $\Lambda_T=B^*$ where $B=[(x^*_{(n,i)})_{(n,i)\in\Lambda}]$. Thus by (7), $B^{**}/B\sim c_0(\Delta)$ ("$\sim$" denotes isomorphism) and so $B^\perp\sim l_1(\Delta)$ ($B^\perp$ taken in B^{***}). Hence

$$\Lambda_T^{**}=B^{***}\sim B^\perp\oplus B^*\sim l_1(\Delta)\oplus\Lambda_T.$$

Alternatively, it is not hard to check directly that if for $\beta\in\Delta$, F_β is the weak*-limit in Λ_T^{**} of the sequence $(x_{(n,i)})_{(n,i)\in\beta}$, then $(F_\beta)_{\beta\in\Delta}$ is 2-equivalent to the unit vector basis of $l_1(\Delta)$ and $\Lambda_T\oplus[(F_\beta)_{\beta\in\Delta}]=\Lambda_T^{**}$.

PROBLEM. Give an example of a nonseparable reflexive space not containing a subsysmmetric basic sequence.

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